

# KHOVANOV'S CONJECTURE OVER $\mathbb{Z}[c]$

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ABSTRACT. We disprove the conjecture by M.Khovanov [K1] on the functoriality of his link homology with polynomial coefficients. This is in contrast to the case of integer coefficients, where the functoriality was proven in [J].

## 1. INTRODUCTION

**1.1. Khovanov's Homology and Conjecture.** In his 1999 paper [K1] in Duke Mathematical Journal, Mikhail Khovanov showed that the Jones link polynomial is the graded Euler characteristic of a bigraded homology module  $H^{i,j}(D)$  over  $\mathbb{Z}[c]$ , associated to a diagram  $D$  of the link.  $H^{i,j}(D)$  is the homology of a bigraded chain complex  $C^{i,j}(D)$  with a  $(1,0)$ -bigraded differential. He also explained how each link cobordism induces a homomorphism between the homology modules of its boundary links, and conjectured that this homomorphism would be invariant up to sign under ambient isotopy of the link cobordism.

In [J] we proved that, if formulated precisely, this conjecture is indeed true, under the condition that the indeterminate  $c = 0$ , i.e. when the modules are abelian groups. (This case suffices to retrieve the Jones polynomial as the Euler characteristic.) In [K2], Khovanov gave a different proof of this fact, and, in the introduction, renewed his conjecture in the general case.

**1.2. The Results of This Paper.** In the present note we consider the complete conjecture, with no apriori conditions on  $c$ . We disprove it by giving an example (Section 3, Figure 9) of a trivial link cobordism from the unlink of two components to itself, which does not induce  $\pm id$  on homology.

**1.3. A Remark on Coefficients.** Throughout this paper, we use coefficients in the polynomial ring (in  $c$ ) over  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ . This only strengthens the result, and simplifies some calculations.

**1.4. Outline.** Section 2 contains necessary preliminaries on Khovanov homology. In Section 3 we present the above-mentioned counterexample to Khovanov's conjecture.

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## 2. PRELIMINARIES

**2.1. Khovanov's Chain Complex.** We briefly review the definition of Khovanov's chain complex  $C^{i,j}(D)$  and its differential. More details may be found in [V] or [J] (and, for the original definitions, in [K1]).

**2.1.1. The Chain Complex.** Let  $D$  be a diagram of an oriented link. Recall that a (Kauffman) state of  $D$  is a distribution of positive or negative *markers* to the crossings of the diagram (see Figure 1). Each marker specifies one of the two ways to smooth the crossing. Smoothing all the crossings according to the markers at them gives a set of embedded circles in the plane called the resolution of  $D$  according to  $S$ .

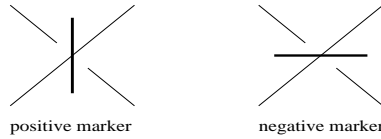


FIGURE 1. Positive and negative markers.

Following Viro [V] we define an *enhanced state*  $S$  to be a state of  $D$ , together with an assignment of the symbols  $\mathbf{1}$  or  $X$  to each circle of the resolution of  $D$  according to  $S$ . (In [V],  $\mathbf{1}$  was denoted by the sign  $-$  and  $X$  by the sign  $+$ .)

*Remark.* Since we will only be concerned with enhanced states, we call them states from now on.

Denote by  $C(D)$  the free  $\mathbb{Z}_2[c]$ -module generated by all (enhanced) states of  $D$ . Denote by  $w(D)$  the writhe of the diagram. Let  $\sigma(S)$  be the sum of all signs of markers in the state  $S$  and let  $\tau(S)$  be the number of  $X$ :s minus the number of  $\mathbf{1}$ :s on the resolution of  $S$ .

$C(D)$  becomes a bigraded module  $C = C^{i,j}(D)$ , if we define the grading parameters for an element  $c^k S$  as

$$i(c^k S) = \frac{w(D) - \sigma(S)}{2}$$

$$j(c^k S) = -\frac{\sigma(S) + 2\tau(S) - 3w(D)}{2} + 2k$$

Notice that multiplication by  $c$  affects only the second grading parameter and that  $\deg(c) = 2$ .

2.1.2. *The Differential.* The differential  $dS$  on a state  $S$  in  $C^{i,j}(D)$  is the sum of all states  $T$  in  $C^{i+1,j}(D)$ , which are *incident* to  $S$  in the following sense.

- The markers of  $T$  coincide with those of  $S$  at all crossings except one, where  $T$  has a negative marker and  $S$  has a positive marker. This means that the resolution of  $T$  has either one circle more or one circle less than the resolution of  $S$ .
- The circles of the resolution which are common to  $S$  and  $T$  are colored with the same symbols in  $S$  as in  $T$ . The circles which are not, have symbols related by the table in Figure 2. It displays resolutions of  $S$  and  $T$  in a neighbourhood of the crossing at which the markers differ, with an indication (dotted arcs) of how the arcs are connected outside this neighbourhood. Note that the fifth row contains three different states incident to  $S$ . They appear in the differential as a sum. The last row is a mere mnemonic, summing up the other rows.
- If  $T$  is incident to  $S$  then  $c^k T$  is incident to  $c^k S$  for all  $k$ .

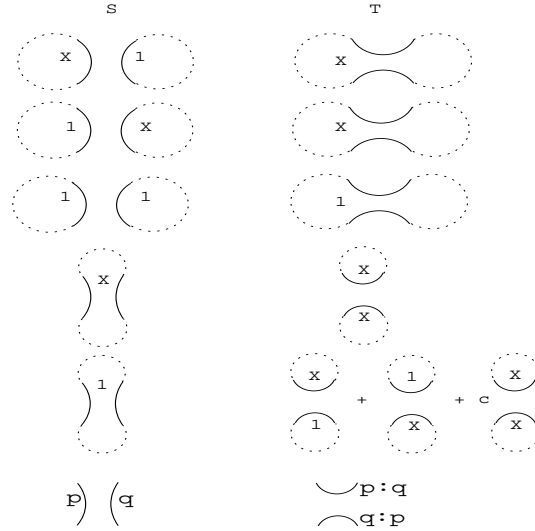


FIGURE 2. Incident states

2.2. **The Second Reidemeister Move  $\Omega_2$ .** In his proof of invariance of the (isomorphism class of) homology groups under Reidemeister moves, Khovanov shows the following proposition. Let  $D'$  be obtained from  $D$  by a Reidemeister move of type 2 increasing the number of crossings. (Reidemeister moves of type 2 will henceforth be referred to as  $\Omega_2$ -moves).

**Proposition 2.1.** *There is a splitting  $C(D') = C \oplus C_{\text{contr}}$ , where  $C_{\text{contr}}$  is chain contractible, and an isomorphism  $\psi : C \xrightarrow{\cong} C(D)$ .*

Let us describe this splitting. Generators of  $C$  and the isomorphism  $C \cong C(D)$  are displayed in Figure 3.  $C_{\text{contr}}$  is generated (as a chain complex) by states as in Figure 4.

*Remark.* The symbols  $p, q, p : q, q : p$  in the figures mean that the symbols on the arcs are related as in Figure 2.

FIGURE 3. The isomorphism  $\psi$ . The expressions on the left generate  $C$ .

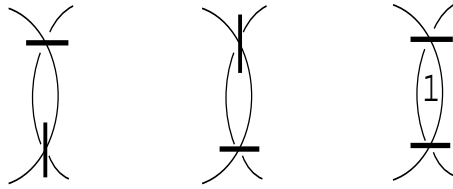


FIGURE 4. Generators of the contractible splitting factor  $C_{contr}$  for the  $\Omega_2$ -move.

It follows that the composition  $\Psi = \psi \circ pr_C$  given by

$$C(D') = C \oplus C_{contr} \xrightarrow{pr_C} C \xrightarrow{\psi} C(D)$$

is a chain equivalence.  $pr_C$  denotes the canonical projection. The inclusion  $i_C$  and the inverse of  $\psi$  give the homotopy inverse  $\Psi^{inv}$ . Using this it is straightforward to prove (using the same method as in Subsection 2.4 in [J]) the following proposition.

**Proposition 2.2.** *The chain equivalences  $\Psi$  and  $\Psi^{inv}$  above have the values given in Figures 6 and 5 on states with given local configuration.*

FIGURE 5. The chain equivalence  $\Psi^{inv}$ .

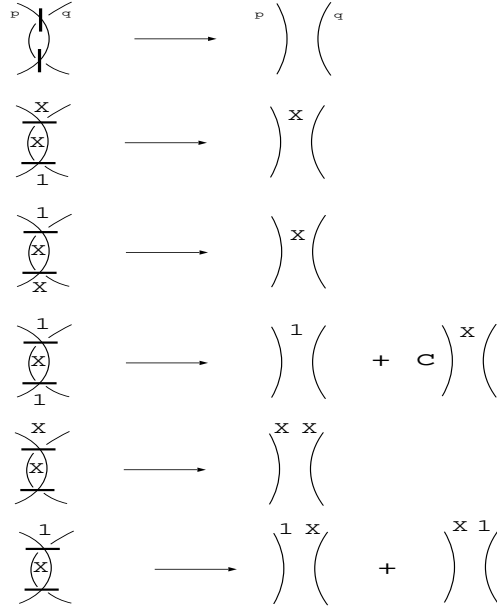


FIGURE 6. The effect of the second Reidemeister move on states. The chain equivalence  $\Psi$ . A state with any other local configuration is mapped to zero.

**2.3. Khovanov's Conjecture.** Recall that a *link cobordism*  $\Sigma \subset \mathbb{R}^3 \times [0, 1]$  between links  $L \subset \mathbb{R}^3 \times \{0\}$  and  $L' \subset \mathbb{R}^3 \times \{1\}$  is a smoothly embedded compact oriented surface whose boundary consists of the links.

Every link cobordism can be presented by a *movie*, which is the sequence of diagrams of the intersection of  $\Sigma$  with the constant time hyperplanes  $\mathbb{R}^3 \times \{t\}$  (subject to certain genericity assumptions). For all but a finite number of values of  $t$  (critical levels), this intersection is a link diagram, and at times just before and after a critical level the diagrams differ by a Reidemeister move or a Morse modification. Between two critical levels the diagram experiences a planar isotopy.

When a link cobordism is altered by an ambient isotopy, its movie changes accordingly. (For additional details on link cobordisms, see e.g. [J].)

In his paper [K1], Khovanov described chain equivalences induced by all Reidemeister moves and chain maps induced by Morse modifications on the link diagram. (In the previous section we explained how this is done for the second Reidemeister move, since this is all we will need in this paper.) It is also clear from the construction that states can be traced through any planar isotopy between two diagrams and hence induces a chain isomorphism between the associated chain complexes.

It follows that each movie presentation of a link cobordism induces a map between the homologies of its boundary link diagrams, via composition.

Khovanov conjectured that this map be invariant up to sign under ambient isotopy of the link cobordism. We pointed out in [J] that it is necessary to require stability of the boundary during the isotopy. (For example, that the boundary is held fixed during the isotopy.) We then proved the conjecture under the assumption that  $c = 0$ .

In the next section we prove that the assumption  $c = 0$  is in fact necessary, by exhibiting a movie presentation of a trivial link cobordism which does not induce the identity on homology.

### 3. A COUNTEREXAMPLE TO KHOVANOV'S CONJECTURE

*Example.* To begin with, let us compute the effect on chains under the following sequence (\*) of  $\Omega_2$ -moves. The sequence starts with a crossing somewhere in a link diagram. Just above the crossing an  $\Omega_2$ -move is performed, creating two bigons. Removal of the lower one of these via an  $\Omega_2$ -move in the opposite direction returns us to the original diagram:

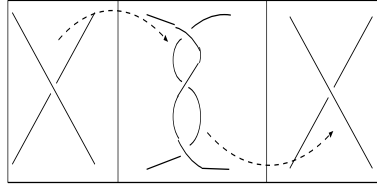


FIGURE 7. The movie (\*).

Figure 8 shows what happens on  $C^{i,j}(D)$ . We include only states whose images we will need later. To verify this table is straightforward, given Proposition 2.2.

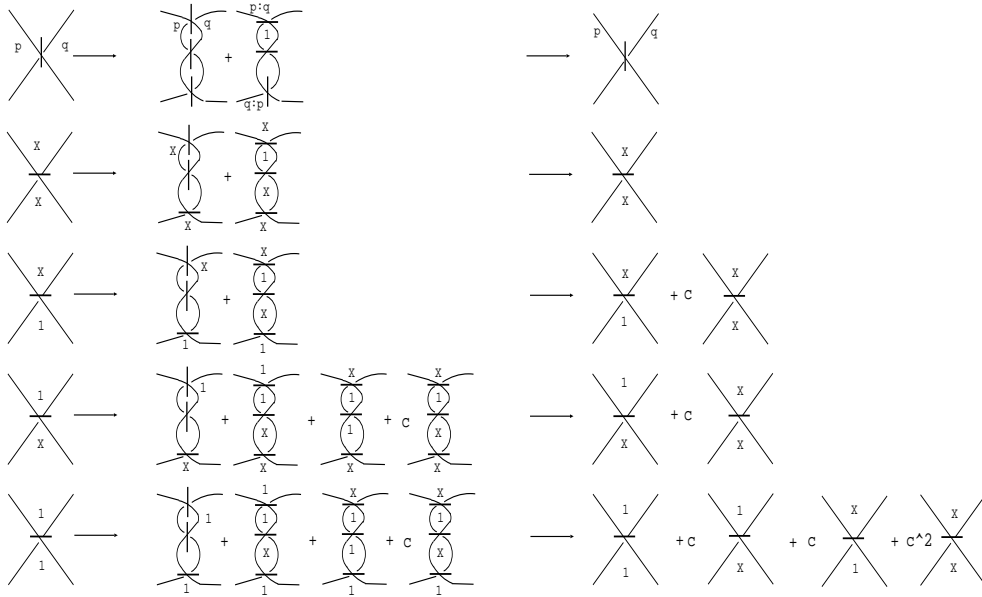


FIGURE 8. The chain map induced by (\*).

Now consider the sequence of moves in Figure 9. It starts and ends with the trivial unlink diagram of two components. Since evidently the differential is zero for this diagram, the homology modules are canonically isomorphic

to the chain modules. Each state is just a distribution of  $1/X$ :s to the components.

The sequence starts with an  $\Omega_2$ -move on the trivial diagram, sliding the left circle above the right one. This induces the chain equivalence  $\Psi^{inv}$  as described in Proposition 2.2. Then the sequence (\*) described above is applied at the upper crossing. Finally the circles slide back again inducing  $\Psi$ .

This is a movie of a link cobordism from the trivial unlink to itself, which is isotopic to the cylinder on the unlink. A simple computation shows that the map induced on the homology of the unlink is the one given in Figure 3. Clearly it is only *id* if  $c = 0$ . This disproves the general version of Khovanov's conjecture.

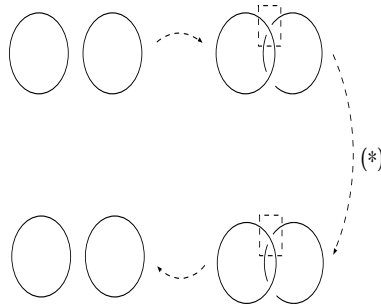


FIGURE 9. A movie of a trivial link cobordism. (\*) refers to the movie in Figure 7 applied in the dotted square.

$$\begin{array}{l}
 \begin{array}{ccc} \textcircled{x} & \textcircled{x} & \longrightarrow \textcircled{x} \textcircled{x} \end{array} \\
 \begin{array}{ccc} \textcircled{x} & \textcircled{1} & \longrightarrow \textcircled{x} \textcircled{1} + c \textcircled{x} \textcircled{x} \end{array} \\
 \begin{array}{ccc} \textcircled{1} & \textcircled{x} & \longrightarrow \textcircled{1} \textcircled{x} + c \textcircled{x} \textcircled{x} \end{array} \\
 \begin{array}{ccccccc} \textcircled{1} & \textcircled{1} & \longrightarrow & \textcircled{1} \textcircled{1} & + c \textcircled{x} \textcircled{1} & + c \textcircled{1} \textcircled{x} & + c^2 \textcircled{x} \textcircled{x} \end{array}
 \end{array}$$

FIGURE 10. The induced map on homology, forcing  $c$  to be 0.

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